

# STANDING WAVES FOR A GAUGED NONLINEAR SCHRÖDINGER EQUATION WITH A VORTEX POINT

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**ABSTRACT.** This paper is motivated by a gauged Schrödinger equation in dimension 2. We are concerned with radial stationary states under the presence of a vortex at the origin. Those states solve a nonlinear nonlocal PDE with a variational structure. We will study the global behavior of that functional, extending known results for the regular case.

## 1. INTRODUCTION

In this paper we are concerned with a planar gauged Nonlinear Schrödinger Equation:

$$(1) \quad iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0.$$

Here  $t \in \mathbb{R}$ ,  $x = (x_1, x_2) \in \mathbb{R}^2$ ,  $\phi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{C}$  is the scalar field,  $A_\mu : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$  are the components of the gauge potential and  $D_\mu = \partial_\mu + iA_\mu$  is the covariant derivative ( $\mu = 0, 1, 2$ ).

The modified gauge field equation proposes the following equation for the gauge potential, including the so-called Chern-Simons term (see [7, 26]):

$$(2) \quad \partial_\mu F^{\mu\nu} + \frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = j^\nu, \quad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

In the above equation,  $\kappa$  is a parameter that measures the strength of the Chern-Simons term. As usual,  $\epsilon^{\nu\alpha\beta}$  is the Levi-Civita tensor, and super-indices are related to the Minkowski metric with signature  $(1, -1, -1)$ . Finally,  $j^\mu$  is the conserved matter current,

$$j^0 = |\phi|^2, \quad j^i = 2\text{Im}(\bar{\phi}D_i\phi).$$

At low energies, the Maxwell term in (2) becomes negligible and can be dropped, giving rise to:

$$(3) \quad \frac{1}{2}\kappa\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = j^\nu.$$

See [9, 10, 14–16] for the discussion above. If we fix  $\kappa = 2$ , equations (1) and (3) lead us to the problem:

$$(4) \quad \begin{aligned} iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi &= 0, \\ \partial_0 A_1 - \partial_1 A_0 &= \text{Im}(\bar{\phi}D_2\phi), \\ \partial_0 A_2 - \partial_2 A_0 &= -\text{Im}(\bar{\phi}D_1\phi), \\ \partial_1 A_2 - \partial_2 A_1 &= \frac{1}{2}|\phi|^2. \end{aligned}$$

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As usual in Chern-Simons theory, problem (4) is invariant under gauge transformation,

$$(5) \quad \phi \rightarrow \phi e^{i\chi}, \quad A_\mu \rightarrow A_\mu - \partial_\mu \chi,$$

for any arbitrary  $C^\infty$  function  $\chi$ .

This model was first proposed and studied in [14–16], and sometimes has received the name of Chern-Simons-Schrödinger equation. The initial value problem, well-posedness, global existence and blow-up, scattering, etc. have been addressed in [2, 11, 13, 20, 21] for the case  $p = 3$ . See also [19] for a global existence result in the defocusing case, and [5] for a uniqueness result to the infinite radial hierarchy.

The existence of stationary states for (4) and general  $p > 1$  has been studied in [3] for the regular case (see also [6, 12, 22, 23]). Very recently, in [4] the case with a vortex point has been considered (with respect to that paper, our notation interchanges the indices 1 and 2). Consider the ansatz:

$$\begin{aligned} \phi &= u(r)e^{i(N\theta + \omega t)}, & A_0 &= A_0(r), \\ A_1 &= -\frac{x_2}{r^2}h(r), & A_2 &= \frac{x_1}{r^2}h(r) \end{aligned}$$

Here  $(r, \theta)$  are the polar coordinates of  $\mathbb{R}^2$ , and  $N \in \mathbb{N} \cup \{0\}$  is the order of the vortex at the origin ( $N = 0$  corresponds to the regular case).

In [4] it is found that  $u$  solves the equation:

$$-\Delta u(x) + \omega u + \frac{(h_u(|x|) - N)^2}{|x|^2} u + A_0(|x|)u(x) = |u(x)|^{p-1}u(x), \quad x \in \mathbb{R}^2,$$

where

$$(6) \quad h_u(r) = \frac{1}{2} \int_0^r s u^2(s) ds,$$

and

$$A_0(r) = \xi + \int_r^{+\infty} \frac{h_u(s) - N}{s} u^2(s) ds, \quad \xi \in \mathbb{R}.$$

The value  $\xi$  above appears as an integration constant. Without loss of generality, we can assume  $\xi = 0$ ; otherwise it suffices to use the gauge invariance (5) with  $\chi = \xi t$ . Then, our problem becomes:

$$(7) \quad -\Delta u(x) + \omega u + \frac{(h_u(|x|) - N)^2}{|x|^2} u + \left( \int_{|x|}^{+\infty} \frac{h_u(s) - N}{s} u^2(s) ds \right) u(x) = |u(x)|^{p-1}u(x).$$

Observe that (7) is a nonlocal equation. In [4] it is shown that (7) is indeed the Euler-Lagrange equation of the energy functional  $I_\omega : \mathcal{H} \rightarrow \mathbb{R}$ ,

$$\begin{aligned} I_\omega(u) &= \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u(x)|^2 + \omega u^2(x)) dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(r) - N)^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^2} |u(x)|^{p+1} dx. \end{aligned}$$

The Hilbert space  $\mathcal{H}$  is defined as:

$$(8) \quad \mathcal{H} = \{u \in H_r^1(\mathbb{R}^2) : \int_{\mathbb{R}} \frac{u^2(x)}{|x|^2} dx < +\infty\},$$

endowed by the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^2} |\nabla u(x)|^2 + \left(1 + \frac{1}{|x|^2}\right) u^2(x) dx.$$

Let us observe that the energy functional  $I_\omega$  presents a competition between the nonlocal term and the local nonlinearity of power-type. The study of the behavior of the functional under this competition is one of the main motivations of this paper. For  $p > 3$ , it is known that  $I_\omega$  is unbounded from below, so it exhibits a mountain-pass geometry (see [3, 12] for the case  $N = 0$  and [4, Section 5] for  $N \in \mathbb{N}$ ). In a certain sense, in this case the local nonlinearity dominates the nonlocal term. However the existence of a solution is not so direct, since for  $p \in (3, 5)$  the (PS) property is not known to hold. This problem is bypassed by combining the so-called monotonicity trick of Struwe ([25]) with a Pohozaev identity.

A special case in the above equation is  $p = 3$ : in this case, solutions have been explicitly found in [3, 4] as optimizers of a certain inequality. An alternative approach would be to pass to a self-dual equation, which leads to a Liouville equation in  $\mathbb{R}^2$ , singular if  $N > 0$ .

The situation is different if  $p \in (1, 3)$ ; here the nonlocal term prevails over the local nonlinearity, in a certain sense. In [22], the second and third authors studied whether  $I_\omega$  is bounded from below or not for  $p \in (1, 3)$  and  $N = 0$ . The situation happened to be quite rich and unexpected, and very different from the usual nonlinear Schrödinger equation. Indeed, the boundedness of  $I_\omega$  for  $N = 0$  depends on the phase  $\omega$  and the threshold value  $\omega_0$  is explicit, namely:

$$(9) \quad \omega_0 = \frac{3-p}{3+p} 3^{\frac{p-1}{2(3-p)}} 2^{\frac{2}{3-p}} \left( \frac{m^2(3+p)}{p-1} \right)^{-\frac{p-1}{2(3-p)}},$$

with

$$(10) \quad m = \int_{-\infty}^{+\infty} \left( \frac{2}{p+1} \cosh^2 \left( \frac{p-1}{2} r \right) \right)^{\frac{2}{1-p}} dr.$$

The purpose of this paper is to extend such result to the case  $N > 0$ , which is more relevant from the point of view of the applications. This study has been prompted by Remark 5.1 in [4].

Our main results are the following:

**Theorem 1.1.** *For  $\omega_0$  as given in (9), there holds:*

- (i) *if  $\omega \in (0, \omega_0)$ , then  $I_\omega$  is unbounded from below;*
- (ii) *if  $\omega = \omega_0$ , then  $I_{\omega_0}$  is bounded from below, not coercive and  $\inf I_{\omega_0} < 0$ ;*
- (iii) *if  $\omega > \omega_0$ , then  $I_\omega$  is bounded from below and coercive.*

Regarding the existence of solutions, we obtain the following results:

**Theorem 1.2.** *There exist  $\bar{\omega} > \tilde{\omega} > \omega_0$  such that:*

- (i) *if  $\omega > \bar{\omega}$ , then (7) has no solutions different from zero;*
- (ii) *if  $\omega \in (\omega_0, \tilde{\omega})$ , then (7) admits at least two positive solutions: one of them is a global minimizer for  $I_\omega$  and the other is a mountain-pass solution;*
- (iii) *for almost every  $\omega \in (0, \omega_0)$  (7) admits a positive solution.*

The proofs follow the same ideas as in [22], and is related to a natural limit problem. Roughly speaking, this limit problem stems from the behavior of the map  $\rho \mapsto I_\omega(u(\cdot - \rho))$  as  $\rho \rightarrow +\infty$ , and this does not depend on  $N$ . However, in our proofs the analysis made in Proposition 3.2 must be re-elaborated with respect to that of [22], and the new terms need new estimates in the asymptotic expansions that follow afterwards. Moreover, the non-existence result of Theorem 1.2 is

immediate for  $N = 0$  but its proof becomes delicate for  $N > 0$ . Finally, the case  $N > 0$  is more relevant from the point of view of the Physics model, since it includes a vortex at the origin. One of the main features of the Chern-Simons theory is the appearance of vortices in the model, see [7, 26, 27].

The rest of the paper is organized as follows. Section 2 is devoted to some notations and preliminary results. In Section 3 we prove Theorems 1.1 and 1.2.

## 2. PRELIMINARIES

Let us first fix some notations. We denote by  $H_r^1(\mathbb{R}^2)$  the Sobolev space of radially symmetric functions, and  $\|\cdot\|$  its usual norm. We denote by  $\|u\|_{L^p}$  the usual Lebesgue norm in  $\mathbb{R}^2$ . Moreover, we will write  $\|\cdot\|_{H^1(\mathbb{R})}$ ,  $\|\cdot\|_{H^1(a,b)}$  to indicate the norms of the Sobolev spaces of dimension 1.

However our functional  $I_\omega$  is defined in the space  $\mathcal{H}$ , defined in (8). Its norm will be denoted by  $\|\cdot\|_{\mathcal{H}}$ . In [4, Proposition 3.1] it is shown that

$$\mathcal{H} \subset \{u \in C(\mathbb{R}^2) : u(0) = 0\} \cap L^\infty(\mathbb{R}^2).$$

If nothing is specified, strong and weak convergence of sequences of functions are assumed in the space  $H^1(\mathbb{R}^2)$ .

In our estimates, we will frequently denote by  $C > 0$ ,  $c > 0$  fixed constants, that may change from line to line, but are always independent of the variable under consideration. We also use the notations  $O(1)$ ,  $o(1)$ ,  $O(\varepsilon)$ ,  $o(\varepsilon)$  to describe the asymptotic behaviors of quantities in a standard way. Finally the letters  $x, y$  indicate two-dimensional variables and  $r, s$  denote one-dimensional variables.

Let us start with the following proposition, proved in [3, 4]:

**Proposition 2.1.**  *$I_\omega$  is a  $C^1$  functional, and its critical points correspond to classical solutions of (7).*

The next result is contained in [4, Proposition 3.4], and deals with the behavior of  $I_\omega$  under weak limits.

**Proposition 2.2.** *Recalling the definition of  $h_u$ , (6), let us define:*

$$(11) \quad K(u) = \frac{1}{2} \int_{\mathbb{R}^2} h_u^2(x) \frac{u^2(x)}{|x|^2} - 2N h_u(x) \frac{u^2(x)}{|x|^2} dx.$$

*Then  $K$  and  $K'$  are weakly continuous in  $\mathcal{H}$ . As a consequence,  $I_\omega$  is weak lower semi-continuous, and  $I'_\omega$  is weakly continuous in  $\mathcal{H}$ .*

Next lemma relates boundedness of sequences in  $H^1(\mathbb{R}^2)$  and in  $\mathcal{H}$ , and will be very useful in Section 3.

**Lemma 2.3.** *The map  $K$  defined in (11) is actually well defined in  $H^1(\mathbb{R}^2)$  and  $K(u_n)$  is bounded if  $\|u_n\|$  is bounded. As a consequence, for any sequence  $u_n \in \mathcal{H}$  such that  $I_\omega(u_n)$  is bounded from above,  $\|u_n\|$  is bounded if and only if  $\|u_n\|_{\mathcal{H}}$  is bounded.*

*Proof.* By [3], we only need to consider the term:

$$\begin{aligned} \int_{\mathbb{R}^2} h_u(x) \frac{u^2(x)}{|x|^2} dx &= \pi \int_0^{+\infty} \frac{u^2(r)}{r} \left( \int_0^r s u^2(s) ds \right) dr \\ &\leq \pi \int_0^{+\infty} u^2(r) \left( \int_0^r u^2(s) ds \right) dr = \frac{\pi}{2} \left( \int_0^{+\infty} u^2(r) dr \right)^2 \end{aligned}$$

Observe now that:

$$2\pi \int_0^{+\infty} u^2(r) dr = \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|} dx \leq \int_{B(0,1)} \frac{u^2(x)}{|x|} dx + \int_{\mathbb{R}^2 \setminus B(0,1)} u^2(x) dx \\ \leq C(\|u\|_{L^2}^2 + \|u\|_{L^p}^2), \quad p > 4,$$

by Holder inequality. The first assertion of the Lemma follows then from the Sobolev embedding.

Suppose that  $u_n$  is bounded in  $H^1(\mathbb{R}^2)$ ; then

$$I_\omega(u_n) = O(1) + \frac{N^2}{2} \int_{\mathbb{R}^2} \frac{u_n^2(x)}{|x|^2} dx,$$

and by hypothesis  $u_n$  is bounded in  $\mathcal{H}$ . The reverse is trivial.  $\square$

The following is a Pohozaev-type identity for problem (7), see (2.11), (5.6) in [4]:

**Proposition 2.4.** *For any  $u \in \mathcal{H}$  solution of (7), the following identity holds:*

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N)^2 dx - \frac{p-1}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} dx = 0.$$

We now state an inequality which will prove to be fundamental in our analysis. This inequality is proved in [4, Proposition 3.5], where also the maximizers are found.

**Proposition 2.5.** *For any  $u \in \mathcal{H}$ ,*

$$(12) \quad \int_{\mathbb{R}^2} |u(x)|^4 dx \leq 4 \left( \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N)^2 dx \right)^{1/2}.$$

As commented in the introduction, this paper is concerned with the boundedness from below of  $I_\omega$ . First of all, let us give a heuristic derivation of the limit energy functional. Consider  $u(r)$  a fixed function, and define  $u_\rho(r) = u(r - \rho)$ . Let us now estimate  $I_\omega(u_\rho)$  as  $\rho \rightarrow +\infty$ ; after the change of variables  $r \rightarrow r + \rho$ , we obtain:

$$\frac{I_\omega(u_\rho)}{2\pi} = \frac{1}{2} \int_{-\rho}^{+\infty} (|u'|^2 + \omega u^2)(r + \rho) dr \\ + \frac{1}{8} \int_{-\rho}^{\infty} \frac{u^2(r)}{r + \rho} \left( \int_{-\rho}^r (s + \rho) u^2(s) ds - 2N \right)^2 dr - \frac{1}{p+1} \int_{-\rho}^{\infty} |u|^{p+1}(r + \rho) dr.$$

We estimate the above expression by simply replacing the expressions  $(r + \rho)$ ,  $(s + \rho)$  with the constant  $\rho$ ; observe that the estimate is independent of  $N$ :

$$(2\pi)^{-1} I_\omega(u_\rho) \\ \sim \rho \left[ \frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2) dr + \frac{1}{8} \int_{-\infty}^{+\infty} u^2(r) \left( \int_{-\infty}^r u^2(s) ds \right)^2 dr - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1} dr \right] \\ = \rho \left[ \frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2) dr + \frac{1}{24} \left( \int_{-\infty}^{+\infty} u^2 dr \right)^3 - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1} dr \right].$$

Therefore, it is natural to consider the limit functional  $J_\omega : H^1(\mathbb{R}) \rightarrow \mathbb{R}$ ,

$$(13) \quad J_\omega(u) = \frac{1}{2} \int_{-\infty}^{+\infty} (|u'|^2 + \omega u^2) dr + \frac{1}{24} \left( \int_{-\infty}^{+\infty} u^2 dr \right)^3 - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1} dr.$$

Clearly, the Euler-Lagrange equation of (13) is the following limit problem:

$$(14) \quad -u'' + \omega u + \frac{1}{4} \left( \int_{-\infty}^{+\infty} u^2(s) ds \right)^2 u = |u|^{p-1} u \quad \text{in } \mathbb{R}.$$

Let  $u$  be a positive solution of (14), and define  $k = \omega + \frac{1}{4} \left( \int_{-\infty}^{+\infty} u^2 dr \right)^2$ . Then, it is well known that  $u(r) = w_k(r - \xi)$  for some  $\xi \in \mathbb{R}$ , where

$$w_k(r) = k^{\frac{1}{p-1}} w_1(\sqrt{k}r), \quad \text{with} \quad w_1(r) = \left( \frac{2}{p+1} \cosh^2 \left( \frac{p-1}{2} r \right) \right)^{\frac{1}{1-p}}.$$

We now recall the value of  $k$ :

$$k = \omega + \frac{1}{4} \left( \int_{-\infty}^{+\infty} w_k^2(r) dr \right)^2 = \omega + \frac{1}{4} k^{\frac{4}{p-1}} \left( \int_{-\infty}^{+\infty} w_1^2(\sqrt{k}r) dr \right)^2.$$

A change of variables leads us to the identity:

$$(15) \quad k = \omega + \frac{1}{4} m^2 k^{\frac{5-p}{p-1}}, \quad k > 0,$$

with  $m$  is given in (10). Therefore, the existence of solutions for (14) reduces to the existence of solutions of the algebraic equation (15). Moreover, we are also interested in the energy of those solutions, and whether it is positive or negative. Those questions have been treated in [22, Section 3], where the following results were obtained:

**Proposition 2.6.** *Assume  $p \in (1, 3)$  and take  $\omega_0$  as in (9). Then:*

- (1) *for any  $\omega > 0$ ,  $J_\omega$  is coercive and attain its infimum;*
- (2) *There exists  $\omega_1 > \omega_0$  such that for  $\omega \in (0, \omega_1)$ , equation (15) has two solutions  $k_1(\omega) < k_2(\omega)$  and  $w_{k_1}(r), w_{k_2}(r)$  are the only two positive solutions of (14) (apart from translations);*
- (3) *if  $\omega > \omega_0$ ,  $\min J_\omega = 0$  and the unique minimizer is 0.*
- (4) *if  $\omega = \omega_0$ ,  $\min J_\omega = 0$  and is attained at 0 and  $w_{k_2}$ .*
- (5) *if  $\omega \in (0, \omega_0)$ ,  $\min J_\omega < 0$  and the minimizer is  $w_{k_2}$ , which is unique (up to change of sign and translation).*

In this paper we are able to relate  $I_\omega$  with the limit functional  $J_\omega$  in the following way:

$$\inf I_\omega > -\infty \Leftrightarrow \inf J_\omega = 0.$$

That is the reason why the explicit value  $\omega_0$  comes as a threshold for  $I_\omega$ .

We finish this section with a technical result from [22, Proposition 3.7], that will be of use later.

**Proposition 2.7.** *Assume  $\omega \geq \omega_0$ , and  $u_n \in H^1(\mathbb{R})$  such that  $J_\omega(u_n) \rightarrow 0$ . There holds*

- (1) *if  $\omega > \omega_0$ , then  $u_n \rightarrow 0$  in  $H^1(\mathbb{R})$ ;*
- (2) *if  $\omega = \omega_0$ , then, up to a subsequence, either  $u_n \rightarrow 0$  or  $u_n(\cdot - x_n) \rightarrow \pm w_{k_2}$  in  $H^1(\mathbb{R})$ , for some sequence  $x_n \in \mathbb{R}$ .*

### 3. PROOF OF THEOREMS 1.1, 1.2

Our first lemma makes rigorous the heuristic derivation of the limit functional made in Section 2. Since the functions in  $\mathcal{H}$  must vanish at 0, we need to truncate our sequence around the origin. For that purpose, take a Lipschitz continuous function  $\phi_0 : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$(16) \quad \phi_0(r) = \begin{cases} 0, & \text{if } |r| \leq 1, \\ 1, & \text{if } |r| \geq 2, \end{cases} \quad |\phi_0'(r)| \leq 1.$$

**Lemma 3.1.** *Let  $U \in H^1(\mathbb{R})$  be an even function which decays to zero exponentially at infinity, and  $\phi_0(r)$  as in (16). Let us denote  $U_\rho(r) = \phi_0(r)U(r - \rho)$ . Then there exists  $C > 0$  such that:*

$$I_\omega(U_\rho) = 2\pi\rho J_\omega(U(r)) - C + o_\rho(1).$$

*Proof.* This estimate has been accomplished in [22, Lemma 4.1] for  $N = 0$ , so we just need to estimate the extra terms:

$$N^2 \int_0^{+\infty} \frac{U_\rho^2(r)}{r} dr - N \int_0^{+\infty} \frac{U_\rho^2(r)}{r} \left( \int_0^r s U_\rho^2(s) ds \right) dr.$$

By using the properties of the cut-off function  $\phi_0$  we have

$$\int_0^{+\infty} \frac{U_\rho^2(r)}{r} dr = o_\rho(1)$$

and it is not difficult to see that

$$\int_0^{+\infty} \frac{U_\rho^2(r)}{r} \left( \int_0^r s U_\rho^2(s) ds \right) dr = \int_{-\infty}^{+\infty} U^2(r) \left( \int_{-\infty}^r U^2(s) ds \right) dr + o_\rho(1) = C + o_\rho(1),$$

with  $C > 0$ . Hence the conclusion follows.  $\square$

In the next proposition we make use of the fundamental inequality (12) to study the behavior of unbounded sequences with energy bounded from above.

**Proposition 3.2.** *Assume  $\omega > 0$ , and  $u_n \in \mathcal{H}$  such that  $\|u_n\|$  is unbounded but  $I_\omega(u_n)$  is bounded from above. Then, there exists a subsequence (still denoted by  $u_n$ ) such that:*

- i) for all  $\varepsilon > 0$ ,  $\int_{\varepsilon \|u_n\|^2}^{+\infty} (|u'_n|^2 + u_n^2) dr \leq C$ ;
- ii) there exists  $\delta \in (0, 1)$  such that  $\int_{\delta \|u_n\|^2}^{\delta^{-1} \|u_n\|^2} (|u'_n|^2 + u_n^2) dr \geq c > 0$ ;
- iii)  $\|u_n\|_{L^2(\mathbb{R}^2)} \rightarrow +\infty$ .

*Proof.* The proof is quite similar to [22, Proposition 4.2], but there are some differences at certain points due to the presence of the singular term. For convenience of the reader, we reproduce it entirely here. By inequality (12) and Cauchy-Schwartz inequality, we can estimate:

$$\begin{aligned} I_\omega(u) &\geq \frac{\pi}{2} \int_0^{+\infty} (|u'|^2 + \omega u^2) r dr + \frac{\pi}{8} \int_0^{+\infty} \frac{u^2(r)}{r} \left( \int_0^r s u^2(s) ds - 2N \right)^2 dr \\ (17) \quad &+ 2\pi \int_0^{+\infty} \left( \frac{\omega}{4} u^2 + \frac{1}{8} u^4 - \frac{1}{p+1} |u|^{p+1} \right) r dr. \end{aligned}$$

Define

$$f : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad f(t) = \frac{\omega}{4} t^2 + \frac{1}{8} t^4 - \frac{1}{p+1} t^{p+1}.$$

Then, the set  $\{t > 0 : f(t) < 0\}$  is of the form  $(\alpha, \beta)$ , where  $\alpha, \beta$  are positive constants depending only on  $p, \omega$ . Moreover, we denote by  $-c_0 = \min f < 0$ .

For each function  $u_n$ , we define:

$$A_n = \{x \in \mathbb{R}^2 : u_n(x) \in (\alpha, \beta)\}, \quad \rho_n = \sup\{|x| : x \in A_n\}.$$

With these definitions, we can rewrite (17) in the form

$$(18) \quad I_\omega(u_n) \geq \frac{\pi}{2} \int_0^{+\infty} (|u'_n|^2 + \omega u_n^2) r dr + \frac{\pi}{8} \int_0^{+\infty} \frac{u_n^2(r)}{r} \left( \int_0^r s u_n^2(s) ds - 2N \right)^2 dr - c_0 |A_n|.$$

In particular this implies that  $|A_n|$  must diverge, and hence  $\rho_n$ . This already proves (iii).

By Strauss Lemma [24], we have

$$(19) \quad \alpha \leq u_n(\rho_n) \leq \frac{\|u_n\|}{\sqrt{\rho_n}}, \Rightarrow \|u_n\|^2 \geq \alpha^2 \rho_n.$$

We now estimate the nonlocal term. For that, define

$$(20) \quad B_n = A_n \cap B(0, \gamma_n), \text{ for } \gamma_n \in (0, \rho_n) \text{ such that } |B_n| = \frac{1}{2}|A_n|.$$

Then  $\int_{B_n} u_n^2(x) dx \geq \alpha^2 |B_n|$  diverges, indeed

$$\int_{B_n} u_n^2(x) dx - 2N > c|A_n|.$$

We now estimate:

$$\begin{aligned} \int_0^{+\infty} \frac{u_n^2(r)}{r} \left( \int_0^r s u_n^2(s) ds - 2N \right)^2 dr &\geq \int_{\gamma_n}^{+\infty} \frac{u_n^2(r)}{r} \left( \int_0^r s u_n^2(s) ds - 2N \right)^2 dr \\ &\geq \frac{1}{4\pi^2} \int_{\gamma_n}^{+\infty} \frac{u_n^2(r)}{r} \left( \int_{B_n} u_n^2(x) dx - 2N \right)^2 dr \\ &\geq c|A_n|^2 \int_{\gamma_n}^{+\infty} \frac{u_n^2(r)}{r} dr \\ &\geq c|A_n|^2 \int_{A_n \setminus B_n} \frac{u_n^2(x)}{|x|^2} dx \\ &\geq c \frac{|A_n|^2}{\rho_n^2} \int_{A_n \setminus B_n} u_n^2(x) dx \\ (21) \quad &\geq c \frac{|A_n|^3}{\rho_n^2}. \end{aligned}$$

Hence, by (17), (19) and (21), we get

$$I_\omega(u_n) \geq c\rho_n + c \frac{|A_n|^3}{\rho_n^2} - c_0|A_n| = \rho_n \left( c + c \frac{|A_n|^3}{\rho_n^3} - c_0 \frac{|A_n|}{\rho_n} \right).$$

Observe that  $t \mapsto c + ct^3 - c_0t$  is strictly positive near zero and goes to  $+\infty$ , as  $t \rightarrow +\infty$ . Then we can assume, passing to a subsequence, that  $|A_n| \sim \rho_n$ . In other words, there exists  $m > 0$  such that  $\rho_n|A_n|^{-1} \rightarrow m$  as  $n \rightarrow +\infty$ .

Taking into account (18) and (19), we conclude that up to a subsequence,  $\|u_n\|^2 \sim \rho_n$ . Moreover, for any fixed  $\varepsilon > 0$ , we have:

$$C\rho_n \geq \|u_n\|_{L^2}^2 \geq \int_{\varepsilon\rho_n}^{+\infty} u_n^2 r dr \geq \varepsilon\rho_n \int_{\varepsilon\rho_n}^{+\infty} u_n^2 dr.$$

An analogous estimate works also for  $\int_{\varepsilon\rho_n}^{+\infty} |u'_n|^2 dr$ . This proves (i).

We now show that for some  $\delta > 0$ ,  $\|u_n\|_{H^1(\delta\rho_n, \rho_n)} \not\rightarrow 0$ , which implies assertion (ii).

First, recall the definition of  $B_n$  and  $\gamma_n$  in (20). Then,

$$\int_{\gamma_n}^{\rho_n} u_n^2(r) dr \geq \rho_n^{-1} \int_{\gamma_n}^{\rho_n} u_n^2(r) r dr \geq \rho_n^{-1} \int_{A_n \setminus B_n} u_n^2(x) dx \geq \rho_n^{-1} |A_n \setminus B_n| \alpha^2 > c > 0.$$

To conclude it suffices to show that  $\gamma_n \sim \rho_n$ . Define

$$C_n = B_n \cap B(0, \tau_n), \text{ for } \tau_n \in (0, \gamma_n) \text{ such that } |C_n| = \frac{1}{2}|B_n|.$$



We can repeat the estimate (21) with  $A_n, B_n$  replaced with  $B_n, C_n$  respectively, to obtain that

$$\int_0^{+\infty} \frac{u_n^2(r)}{r} \left( \int_0^r s u_n^2(s) ds - 2N \right)^2 dr \geq c \frac{|B_n|^3}{\gamma_n^2}.$$

Hence,

$$I_\omega(u_n) \geq c\rho_n + c \frac{|A_n|^3}{\gamma_n^2} - c_0|A_n| = \gamma_n \left( c \frac{\rho_n}{\gamma_n} + c \frac{|A_n|^3}{\gamma_n^3} - c_0 \frac{|A_n|}{\gamma_n} \right).$$

And we are done since  $I_\omega(u_n)$  is bounded from above.  $\square$

*Proof of Theorem 1.1.* If  $\omega \in (0, \omega_0)$ , then  $J_\omega(w_{k_2}) < 0$  (see Proposition 2.6): applying Lemma 3.1 with  $U = w_{k_2}$  we conclude assertion (i).

We now prove (ii) and (iii). We denote by  $H_{0,r}^1(B(0, R))$  the Sobolev space of radial functions with zero boundary value and

$$\mathcal{H}(B(0, R)) = \left\{ u \in H_{0,r}^1(B(0, R)) : \int_{B(0, R)} \frac{u^2(x)}{|x|^2} dx < +\infty \right\},$$

endowed by the norm  $\|\cdot\|_{\mathcal{H}}$ .

Fixed  $n \in \mathbb{N}$  and given a sequence  $v_i \in \mathcal{H}(B(0, n))$  unbounded with respect to the norm  $\|\cdot\|$ , (18) implies that  $I_\omega(v_i) \rightarrow +\infty$ . By Lemma 2.3, we conclude that  $I_\omega|_{\mathcal{H}(B(0, n))}$  is coercive.

So, there exists  $u_n$  a minimizer for  $I_\omega|_{\mathcal{H}(B(0, n))}$ . By taking absolute value, we can assume that  $u_n \geq 0$ . Moreover,

$$I_\omega(u_n) \rightarrow \inf I_\omega, \text{ as } n \rightarrow +\infty.$$

In the following,  $u_n$  may be extended as functions in  $\mathcal{H}$  by setting  $u_n(x) = 0$  for  $x \in \mathbb{R}^2 \setminus B(0, n)$ . If  $u_n$  is bounded in  $H^1(\mathbb{R}^2)$ , Lemma 2.3 implies that  $u_n$  is bounded in  $\mathcal{H}$  and then  $I_\omega(u_n)$  is bounded. In such case we conclude that  $\inf I_\omega$  is finite. In what follows we assume that  $u_n$  is an unbounded sequence in  $H^1(\mathbb{R}^2)$ , and we shall show that  $I_\omega(u_n)$  is still bounded for  $\omega \geq \omega_0$ .

Our sequence  $u_n$  satisfies the hypotheses of Proposition 3.2, so let  $\delta > 0$  be given by that proposition.

The proof will be divided in several steps.

**Step 1:**

$$\int_{\frac{\delta}{2}\|u_n\|^2}^{\frac{2}{\delta}\|u_n\|^2} |u_n|^{p+1} dr \rightarrow 0.$$

By Proposition 3.2, i), we have that:

$$\sum_{k=1}^{\lceil \frac{\delta}{2}\|u_n\|^2 \rceil} \int_{\frac{\delta}{2}\|u_n\|^2+k-1}^{\frac{\delta}{2}\|u_n\|^2+k} (|u_n'|^2 + u_n^2) dr \leq \int_{\frac{\delta}{2}\|u_n\|^2}^{\delta\|u_n\|^2} (|u_n'|^2 + u_n^2) dr \leq C.$$

Taking the smaller summand in the left hand side we find  $x_n$ ,

$$\frac{\delta}{2}\|u_n\|^2 \leq x_n \leq \delta\|u_n\|^2 - 1 \text{ such that } \|u_n\|_{H^1(x_n, x_n+1)}^2 \leq \frac{C}{\|u_n\|^2}.$$

Reasoning in an analogous way, we can choose  $y_n$ ,

$$\delta^{-1}\|u_n\|^2 + 1 \leq y_n \leq 2\delta^{-1}\|u_n\|^2 \text{ such that } \|u_n\|_{H^1(y_n, y_n+1)}^2 \leq \frac{C}{\|u_n\|^2}.$$

Observe that if  $\delta^{-1}\|u_n\|^2 \geq n$ , the choice of  $y_n$  can be arbitrary, but it is unnecessary. Take  $\phi_n : [0, +\infty] \rightarrow [0, 1]$  be a  $C^\infty$ -function such that

$$\phi_n(r) = \begin{cases} 0, & \text{if } r \leq x_n, \\ 1, & \text{if } x_n + 1 \leq r \leq y_n, \quad |\phi'_n(r)| \leq 2. \\ 0, & \text{if } r \geq y_n + 1. \end{cases}$$

Let

$$F(u) = \int_0^{+\infty} \frac{u^2(r)}{r} \left( \int_0^r su^2(s)ds - 2N \right)^2 dr.$$

By the choice of  $x_n, y_n$  and Proposition 3.2, i), we have

$$\begin{aligned} F'(u_n)[\phi_n u_n] &\geq 4 \int_0^{+\infty} \frac{u_n^2(r)}{r} \left( \int_0^r su_n^2(s)ds - 2N \right) \left( \int_0^r su_n^2(s)\phi_n(s)ds \right) dr \\ &\geq -8N \left( \int_{x_n}^{+\infty} u_n^2(r)dr \right)^2 > -C. \end{aligned}$$

It follows that

$$\begin{aligned} 0 = I'_\omega(u_n)[\phi_n u_n] &\geq 2\pi \int_{x_n}^{y_n} (|u'_n|^2 + \omega u_n^2) r dr - 2\pi \int_{x_n}^{y_n} |u_n|^{p+1} r dr + O(1) \\ &\geq \|u_n\|^2 \left( \frac{\delta}{2} \int_{x_n}^{y_n} (|u'_n|^2 + \omega u_n^2) dr - \frac{2}{\delta} \int_{x_n}^{y_n} |u_n|^{p+1} dr \right) + O(1). \end{aligned}$$

This, together with the fact that  $\|u_n\|_{H^1(x_n, y_n)}$  does not tend to zero, allows us to conclude the proof of Step 1.

**Step 2:** Exponential decay.

At this point we can apply the concentration-compactness principle (see [18, Lemma 1.1]); there exists  $\sigma > 0$  such that

$$\sup_{\xi \in [x_n, y_n]} \int_{\xi-1}^{\xi+1} u_n^2 dr \geq 2\sigma > 0.$$

Let us define:

$$(22) \quad D_n = \left\{ \xi > 0 : \int_{\xi-1}^{\xi+1} (|u'_n|^2 + u_n^2) dr \geq \sigma \right\} \neq \emptyset, \text{ and } \xi_n = \max D_n \in [x_n, n+1).$$

Let us observe that  $\xi_n \sim \|u_n\|^2$ ; indeed  $\xi_n \geq x_n \geq c\|u_n\|^2$  and, moreover,

$$\|u_n\|^2 \geq c \int_{\xi_n-1}^{\xi_n+1} (|u'_n|^2 + u_n^2) r dr \geq c(\xi_n - 1) \int_{\xi_n-1}^{\xi_n+1} (|u'_n|^2 + u_n^2) dr \geq c(\xi_n - 1).$$

By definition,  $\int_{\xi-1}^{\xi+1} (|u'_n|^2 + u_n^2) dr < \sigma$  for all  $\zeta > \xi_n$ . By embedding of  $H^1(\zeta - 1, \zeta + 1)$  in  $L^\infty$ ,  $0 \leq u_n(\zeta) < C\sqrt{\sigma}$  for any  $\zeta > \xi_n$ . From this we will get exponential decay of  $u_n$ . Indeed,  $u_n$  is a solution of

$$-u_n''(r) - \frac{u_n'(r)}{r} + \omega u_n(r) + f_n(r)u_n(r) = u_n^p(r),$$

with

$$f_n(r) = \frac{(h_n(r) - N)^2}{r^2} + \int_r^n \frac{h_n(s) - N}{s} u_n^2(s) ds, \quad h_n(r) = \frac{1}{2} \int_0^r su_n^2(s) ds.$$

If  $r > \delta \|u_n\|^2$ , again by Proposition 3.2, i), we see that  $\int_r^n \frac{N}{s} u_n^2(s) ds = o(1)$ . Then, by taking smaller  $\sigma$ , if necessary, we can conclude that there exists  $C > 0$  such that

$$|u_n(r)| < C \exp \left( -\sqrt{\frac{\omega}{2}}(r - \xi_n) \right), \quad \text{for all } r > \xi_n.$$

The local  $C^1$  regularity theory for the Laplace operator (see [8, Section 3.4]) implies a similar estimate for  $u'_n(r)$ . In other words,

$$|u_n(r)| + |u'_n(r)| < C \exp \left( -\sqrt{\frac{\omega}{2}}(r - \xi_n) \right), \quad \text{for all } r > \xi_n.$$

**Step 3: Splitting of  $I_\omega(u_n)$ .**

Reasoning as in the beginning of Step 1, we can take  $z_n$ :

$$\xi_n - 3\|u_n\| \leq z_n \leq \xi_n - 2\|u_n\| \quad \text{with} \quad \|u_n\|_{H^1(z_n, z_n+1)}^2 \leq \frac{C}{\|u_n\|}.$$

Define  $\psi_n : [0, +\infty] \rightarrow [0, 1]$  be a smooth function such that

$$\psi_n(r) = \begin{cases} 0, & \text{if } r \leq z_n, \\ 1, & \text{if } r \geq z_n + 1, \end{cases} \quad |\psi'_n(r)| \leq 2.$$

We claim that

$$(23) \quad I_\omega(u_n) \geq I_\omega(u_n \psi_n) + I_\omega(u_n(1 - \psi_n)) + c\|u_n(1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(\|u_n\|).$$

This estimate has been accomplished in [22] for  $N = 0$ . Therefore we just need to estimate the two new terms; it is easy to get that

$$\int_0^n \frac{u_n^2(r)}{r} dr = \int_0^n \frac{u_n^2(r) \psi_n^2(r)}{r} dr + \int_0^n \frac{u_n^2(r) (1 - \psi_n(r))^2}{r} dr + o(1).$$

Moreover,

$$\begin{aligned} \int_0^n \frac{u_n^2(r)}{r} \left( \int_0^r s u_n^2(s) ds \right) dr &= \int_0^n \frac{u_n^2(r) \psi_n^2(r)}{r} \left( \int_0^r s \psi_n^2(s) u_n^2(s) ds \right) dr \\ &\quad + \int_0^n \frac{u_n^2(r) (1 - \psi_n(r))^2}{r} \left( \int_0^r s (1 - \psi_n(s))^2 u_n^2(s) ds \right) dr \\ &\quad + \underbrace{\int_0^n \frac{u_n^2(r) \psi_n^2(r)}{r} \left( \int_0^r s u_n^2(s) (1 - \psi_n^2(s)) ds \right) dr}_{(I)} \\ &\quad + 2 \underbrace{\int_0^n \frac{u_n^2(r) \psi_n^2(r)}{r} \left( \int_0^r s u_n^2(s) \psi_n(s) (1 - \psi_n(s)) ds \right) dr}_{(II)} \\ &\quad + \underbrace{\int_0^n \frac{u_n^2(r) (1 - \psi_n(r))^2}{r} \left( \int_0^r s u_n^2(s) \psi_n^2(s) ds \right) dr}_{(III)} \\ &\quad + 2 \underbrace{\int_0^n \frac{u_n^2(r) (1 - \psi_n(r))^2}{r} \left( \int_0^r s u_n^2(s) \psi_n(s) (1 - \psi_n(s)) ds \right) dr}_{(IV)} \\ &\quad + \underbrace{\int_0^n \frac{u_n^2(r) (1 - \psi_n(r)) \psi_n(r)}{r} \left( \int_0^r s u_n^2(s) ds \right) dr}_{(V)}. \end{aligned}$$

We now observe that (I) ... (V) are bounded, as follows:

$$(I) \leq \int_{z_n}^n \frac{u_n^2(r)}{r} \left( \int_0^{z_n+1} s u_n^2(s) ds \right) dr \leq \frac{C \|u_n\|^2}{z_n} = O(1),$$

$$(II) \leq 2 \int_{z_n}^n \frac{u_n^2(r)}{r} \left( \int_{z_n}^{z_n+1} s u_n^2(s) ds \right) dr = O(1),$$

and the other terms can be estimated similarly. Therefore, we conclude the proof of (23).

**Step 4:** The following estimate holds:

$$(24) \quad I_\omega(u_n \psi_n) = 2\pi \xi_n J_\omega(u_n \psi_n) + O(\|u_n\|).$$

In [22] this estimate was made for  $N = 0$ . So we just need to check the new nonlocal terms

$$\begin{aligned} \int_0^n \frac{(u_n \psi_n)^2(r)}{r} dr &\leq \frac{C}{z_n} = o(1), \\ \int_0^n \frac{(u_n \psi_n)^2(r)}{r} \left( \int_0^r s (u_n \psi_n)^2(s) ds \right) dr &\leq \int_{z_n}^n \frac{u_n^2(r)}{r} \left( \int_{z_n}^r s u_n^2(s) ds \right) dr \\ &\leq \left( \int_{z_n}^n u_n^2(r) dr \right)^2 = O(1). \end{aligned}$$

**Step 5:** Conclusion for  $\omega > \omega_0$ .

By (23) and (24), we have

$$(25) \quad I_\omega(u_n) \geq 2\pi \xi_n J_\omega(u_n \psi_n) + I_\omega(u_n(1 - \psi_n)) + c \|u_n(1 - \psi_n)\|_{L^2(\mathbb{R}^2)}^2 + O(\|u_n\|).$$

Recall that  $\|u_n \psi_n\|_{H^1(\mathbb{R})}^2 \geq \sigma > 0$ . By Proposition 2.7, we have that  $J_\omega(u_n \psi_n) \rightarrow c > 0$ , up to a subsequence. Since  $\xi_n \sim \|u_n\|^2$ , it turns out from (25) that  $I_\omega(u_n) > I_\omega(u_n(1 - \psi_n))$ , which is a contradiction with the definition of  $u_n$ . Therefore,  $u_n$  needs to be a bounded sequence and, in particular,  $\inf I_\omega > -\infty$ .

Let us now show that  $I_\omega$  is coercive. Indeed, take  $u_n \in \mathcal{H}$  an unbounded sequence, and assume that  $I_\omega(u_n)$  is bounded from above. By Lemma 2.3,  $\|u_n\|$  is unbounded, so that Proposition 3.2, (iii), shows us that  $I_{\hat{\omega}}(u_n) \rightarrow -\infty$  for any  $\omega_0 < \hat{\omega} < \omega$ , a contradiction.

**Step 6:** Conclusion for  $\omega = \omega_0$ .

As above, (25) gives a contradiction unless  $J_\omega(u_n \psi_n) \rightarrow 0$ . Proposition 2.7 now implies that  $\psi_n u_n(\cdot - t_n) \rightarrow w_{k_2}$  up to a subsequence, for some  $t_n \in (0, +\infty)$ . Since  $\xi_n \in D_n$  (recall its definition in (22)), we have that  $|t_n - \xi_n|$  is bounded. With this extra information, we have a better estimate of the decay of the solutions: indeed,

$$(26) \quad |u_n(r)| + |u_n'(r)| < C \exp\left(-\sqrt{\frac{\omega}{2}}|r - \xi_n|\right), \quad \text{for all } r > \xi_n - 2\|u_n\|.$$

This allows us to do the cut-off procedure in a much more accurate way. Indeed, take  $\tilde{z}_n = \xi_n - \|u_n\|$ . Then, (26) implies that

$$(27) \quad \|u_n\|_{H^1(\tilde{z}_n, \tilde{z}_n+1)}^2 \leq C \exp\left(-\sqrt{\frac{\omega}{2}}\|u_n\|\right).$$

Define  $\tilde{\psi}_n : [0, +\infty] \rightarrow [0, 1]$  accordingly:

$$\tilde{\psi}_n(r) = \begin{cases} 0, & \text{if } r \leq \tilde{z}_n, \\ 1, & \text{if } r \geq \tilde{z}_n + 1, \end{cases} \quad |\tilde{\psi}_n'(r)| \leq 2.$$

The advantage is that, in the estimate of  $I_\omega(u_n)$ , now the errors are exponentially small. Indeed, by repeating the estimates of Step 3 with the new information (27), we obtain:

$$I_\omega(u_n) \geq I_\omega(u_n \tilde{\psi}_n) + I_\omega(u_n(1 - \tilde{\psi}_n)) + c\|u_n(1 - \tilde{\psi}_n)\|_{L^2(\mathbb{R}^2)}^2 + O(1),$$

Then,

$$\begin{aligned} I_\omega(u_n) &\geq I_\omega(u_n \tilde{\psi}_n) + I_\omega(u_n(1 - \tilde{\psi}_n)) + c\|u_n(1 - \tilde{\psi}_n)\|_{L^2(\mathbb{R}^2)}^2 + O(1) \\ &= 2\pi\xi_n J_\omega(u_n \tilde{\psi}_n) + I_\omega(u_n(1 - \tilde{\psi}_n)) + c\|u_n(1 - \tilde{\psi}_n)\|_{L^2(\mathbb{R}^2)}^2 + O(1) \\ &\geq I_{(\omega+2c)}(u_n(1 - \tilde{\psi}_n)) + O(1). \end{aligned}$$

But, by Step 5, we already know that  $I_{(\omega+2c)}$  is bounded from below, and hence  $\inf I_{\omega_0} > -\infty$ .

Finally, by applying Lemma 3.1 to  $U = w_{k_2}$  we readily get that  $I_{\omega_0}$  is not coercive.  $\square$

*Proof of Theorem 1.2.* We shall prove each assessment separately.

**Proof of (ii).** First, we observe that since  $\inf I_{\omega_0} < 0$ , there exists  $\tilde{\omega} > \omega_0$  such that  $\inf I_\omega < 0$  if and only if  $\omega \in (\omega_0, \tilde{\omega})$ . Since, by Theorem 1.1 and Proposition 2.2,  $I_\omega$  is coercive and weakly lower semicontinuous, we infer that the infimum is attained at a negative value. This gives the first solution  $u_1$ .

Clearly, 0 is a local minimum for  $I_\omega$ , and  $I_\omega(u_1) < 0$ . Then, the functional satisfies the geometrical assumptions of the Mountain Pass Theorem, see [1]. Since  $I_\omega$  is coercive, (PS) sequences are bounded. By the compact embedding of  $H_r^1(\mathbb{R}^2)$  into  $L^{p+1}(\mathbb{R}^2)$  and Proposition 2.2, standard arguments show that  $I_\omega$  satisfies the Palais-Smale condition and so we find a second solution which is at a positive energy level.

**Proof of (iii).** Let now consider  $\omega \in (0, \omega_0)$ . Performing the rescaling  $u \mapsto u_\omega = \sqrt{\omega} u(\sqrt{\omega} \cdot)$ , we get

$$\begin{aligned} I_\omega(u_\omega) &= \omega \left[ \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) dx + \frac{1}{8} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left( \int_0^{|x|} s u^2(s) ds - 2N \right)^2 dx \right. \\ &\quad \left. - \frac{\omega^{\frac{p-3}{2}}}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} dx \right]. \end{aligned}$$

Define  $\lambda = \omega^{\frac{p-3}{2}}$  and  $\mathcal{I}_\lambda : H(\mathbb{R}^2) \rightarrow \mathbb{R}$  as

$$\mathcal{I}_\lambda(u) = \Phi(u) - \frac{\lambda}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} dx,$$

with

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N)^2 dx \\ &= \frac{1}{2}\|u\|^2 + K(u) + \frac{N^2}{2} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} dx,\end{aligned}$$

where  $K$  is as defined in (11). Then  $\mathcal{I}_\lambda$  satisfies the geometrical assumptions of the Mountain Pass Theorem. The main problem here is that we do not know whether a (PS) sequence could be unbounded.

By Lemma 2.3, the functional  $\Phi : \mathcal{H} \rightarrow \mathbb{R}$  is coercive. Then we can use [17, Theorem 1.1] to obtain a bounded Palais-Smale sequence  $u_n \in \mathcal{H}$  for almost every  $\lambda$ . Passing to a subsequence, we can assume that  $u_n \rightharpoonup u$ ; Proposition 2.2 and standard arguments imply that  $u$  is a critical point of  $\mathcal{I}_\lambda$ . Making the change of variables back we obtain a solution of (7) for almost every  $\omega \in (0, \omega_0)$ .

Finally, in order to find positive solutions of (7), we simply observe that the above arguments apply to the functional  $I_\omega^+ : \mathcal{H} \rightarrow \mathbb{R}$

$$\begin{aligned}I_\omega^+(u) &= \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2) dx + \frac{1}{8} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left( \int_0^{|x|} s u^2(s) ds - 2N \right)^2 dx \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}^2} (u^+)^{p+1} dx.\end{aligned}$$

Due to the maximum principle, the critical points of  $I_\omega^+$  are positive solutions of (7).

**Proof of (i).** This part happens to be quite delicate, compared to the case  $N = 0$  studied in [22]. Let  $u$  be a solution of (7). If we multiply (7) by  $u$  and integrate, we get

$$\begin{aligned}0 &= \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2) dx + 3 \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N)^2 dx \\ (28) \quad &\quad + 2N \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N) dx - \int_{\mathbb{R}^2} |u|^{p+1} dx.\end{aligned}$$

From (28) and the Pohozaev identity (Proposition 2.4), we obtain that, for any  $l > 0$ ,

$$\begin{aligned}0 &= (l+1) \int_{\mathbb{R}^2} |\nabla u|^2 dx + \omega \int_{\mathbb{R}^2} u^2 dx + (l+3) \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N)^2 dx \\ (29) \quad &\quad + 2N \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N) dx - \left( \frac{p-1}{p+1} l + 1 \right) \int_{\mathbb{R}^2} |u|^{p+1} dx.\end{aligned}$$

By using (12) in (29),

$$\begin{aligned}0 &\geq \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2) dx + 3 \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N)^2 dx \\ (30) \quad &\quad + 2N \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N) dx - \left( \frac{p-1}{p+1} l + 1 \right) \int_{\mathbb{R}^2} |u|^{p+1} dx + \frac{l}{2} \int_{\mathbb{R}^2} |u|^4 dx.\end{aligned}$$

We can estimate

$$\begin{aligned}
 & 3 \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N)^2 + 2N \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N) dx \\
 &= \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} (h_u(|x|) - N)(3h_u(|x|) - N) dx \\
 (31) \quad &\geq -\frac{N^2}{3} \int_{\{N/3 \leq h_u \leq N\}} \frac{u^2(x)}{|x|^2} dx,
 \end{aligned}$$

where  $\{N/3 \leq h_u \leq N\} = \{r_0 \leq |x| \leq r_1\}$  with  $h_u(r_0) = N/3$  and  $h_u(r_1) = N$  (here we have used that  $h_u$  is increasing in  $r$ ). For any  $r > 0$ , by the definition of  $h_u$  we have

$$4\pi h_u(r) = \int_{B_r} u^2(x) dx \leq Cr \left( \int_{B_r} u^4(x) dx \right)^{1/2}.$$

Then

$$\begin{aligned}
 (32) \quad & \int_{B_r} h_u^2(|x|) \frac{u^2(x)}{|x|^2} dx \leq C \int_{B_r} u^2(x) \left( \int_{B_{|x|}} u^4(y) dy \right) dx \\
 & \leq C \int_{B_r} u^2(x) dx \int_{B_r} u^4(x) dx \leq Ch_u(r) \int_{B_r} u^4(x) dx.
 \end{aligned}$$

We now apply (32) to estimate

$$\begin{aligned}
 & \int_{\{N/3 \leq h_u \leq N\}} \frac{u^2(x)}{|x|^2} dx \leq C \int_{\{N/3 \leq h_u \leq N\}} h_u^2(|x|) \frac{u^2(x)}{|x|^2} dx \leq C \int_{B_{r_1}} h_u^2(|x|) \frac{u^2(x)}{|x|^2} dx \\
 (33) \quad & \leq Ch_u(r_1) \int_{B_{r_1}} u^4(x) dx \leq C \int_{\mathbb{R}^2} u^4 dx.
 \end{aligned}$$

We apply (31) and (33) in (30):

$$0 \geq \int_{\mathbb{R}^2} |\nabla u|^2 dx + \omega \int_{\mathbb{R}^2} u^2 dx - c \int_{\mathbb{R}^2} u^4 dx + \frac{l}{2} \int_{\mathbb{R}^2} u^4 - \left( \frac{p-1}{p+1} l + 1 \right) \int_{\mathbb{R}^2} |u|^{p+1} dx.$$

Therefore it suffices to take  $l$  so that  $-c + \frac{l}{2} = 1$ , and then to take  $\omega$  so that the function

$$s \rightarrow \omega s^2 + s^4 - \left( \frac{p-1}{p+1} l + 1 \right) |s|^{p+1}$$

is non-negative for any  $s$ . Therefore  $u$  must be identically equal to zero.  $\square$

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